An Overview of Relative Trisections

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First the closed case ...

Def (Gay–Kirby 12) a (g,k)-trisection of a closed smooth 4-mfd X is a decomposition $X = X_1 \cup X_2 \cup X_3$ where

$$X_i \cong \natural^{k} S^1 \times B^3$$

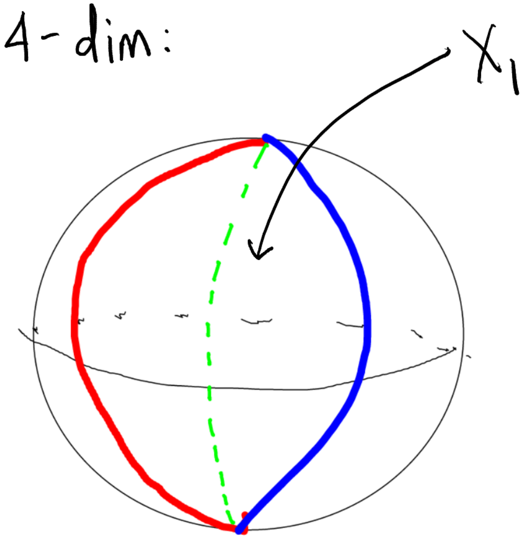
$$\partial X_i \cong \#^{k} S^1 \times S^2$$
$$= (X_i \cap X_{i-1}) \cup (X_i \cap X_{i+1})$$

a Heegaard splitting of ∂X_i:

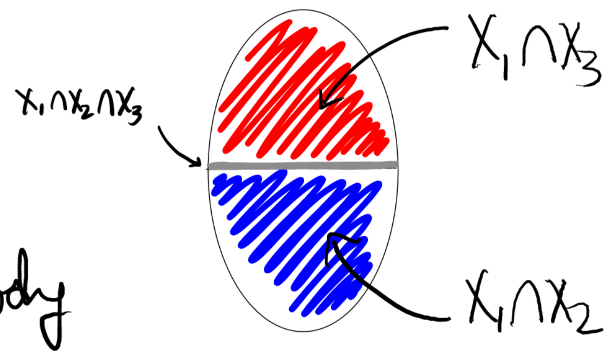
$X_i \cap X_{i-1} \cong X_i \cap X_{i+1} \cong H_g$ genus g handlebody

$X_1 \cap X_2 \cap X_3 \cong S_g$ genus g surface

4-dim:



3-dim:



Def (Gay-Kirby) A (g,k)-trisection diagram is a tuple $(S_g, \alpha, \beta, \gamma)$ such that

$$\left.\begin{array}{l}(S_g, \alpha, \beta)\\ (S_g, \beta, \gamma)\\ (S_g, \gamma, \alpha)\end{array}\right\}\ \text{are all Heegaard diagrams of } \#^k S^1 \times S^2$$

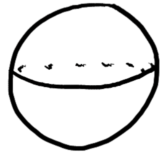
Thm (Gay-Kirby)

A. Trisected 4-mfds / diffeo $\xleftrightarrow{1-1}$ Trisection diag / diffeo, handle slides, isotopy

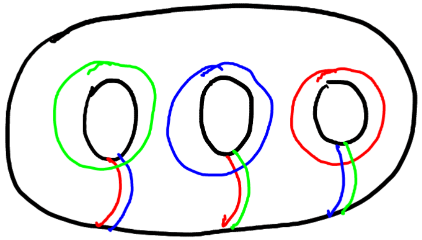
B. 4-mfds / diffeo $\xleftrightarrow{1-1}$ Trisection diag / ″ stabilization

Examples:

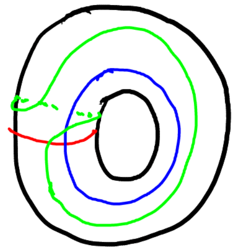
S^4 $(S^2, \emptyset, \emptyset, \emptyset)$



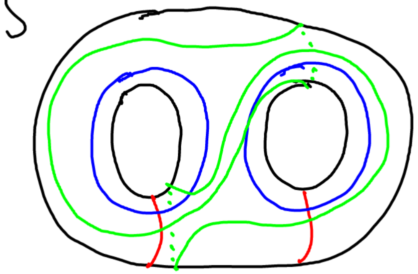
S^4



$\mathbb{C}P^2$



$S^2 \times S^2$



Trisections of manifolds with bdry

Def (A modification of Gay-Kirby) A relative trisection of a 4-mfd X with $\partial X \neq \emptyset$, ∂X connected

is a decomposition $X = X_1 \cup X_2 \cup X_3$

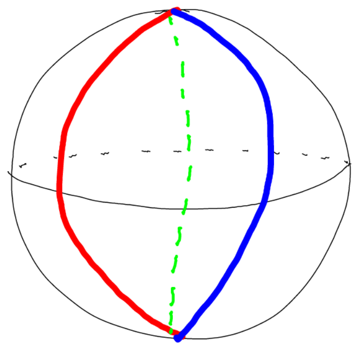
$$X_i \cong \natural^k S^1 \times B^3$$

$$\partial X_i \cong \#^k S^1 \times S^2 = (X_i \cap X_{i-1}) \cup (X_i \cap X_{i+1}) \cup (X_i \cap \partial X_i)$$

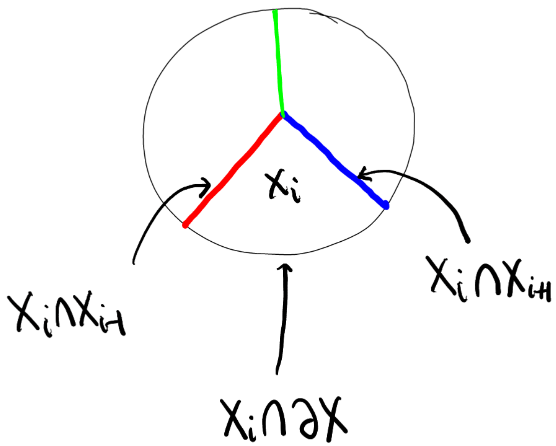
such that ...

(Need new terminology)

Closed case:



Relative case:



Sutured manifolds (A short detour)

Def A sutured mfd is an oriented 3-mfd Y with a decomposition

vertical part

$$\partial Y = R_- \cup \Gamma \cup R_+$$

horizontal part

(i) Every component of Γ is either T^2 or $S^1 \times I$

(ii) $\partial R_- \cap \partial R_+ = \emptyset$

Two meaningful examples:

A. P a compact oriented surface with bdry. Df

$$N = P \times I$$
$$\Gamma = \partial P \times I$$
$$R_\mp = P \times \{\mp 1\},$$

then $\boxed{N(P) = (N, \Gamma)}$ is a sutured mfd.

B. P as before, Y a 3-mfd $P \subseteq Y$. Df

$$M = Y \setminus \mathrm{Int}(P \times I)$$
$$\Gamma = \partial P \times I$$

then $\boxed{Y(P) = (M, \Gamma)}$ is a sutured manifold.

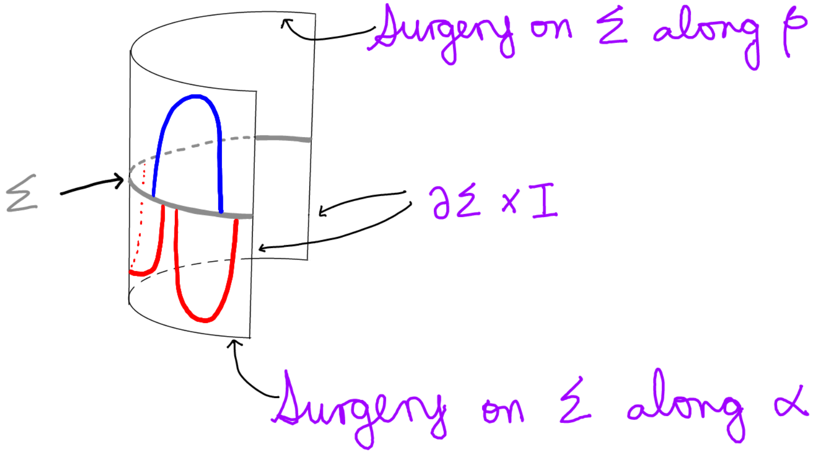
Def (Juhasz) a sutured Heegaard diagram is a tuple (Σ, α, β) where
- Σ is a surface
- α, β sets of simple closed curves.

Thm (Juhasz)

$$\text{Sutured manifolds}/\text{diffeo} \xleftrightarrow{1-1} \text{Sutured diagrams}/\begin{matrix}\text{surface diffeo}\\ \text{isotopy}\\ \text{handle slides}\end{matrix}$$

Pf (sketch)

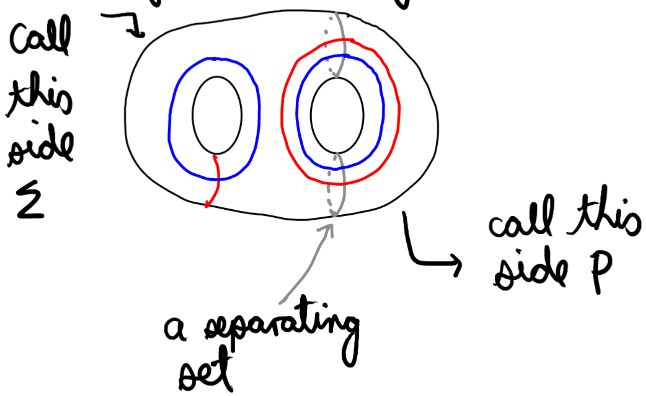
$$M = (\Sigma \times I) \cup \begin{matrix}\text{2-handles}\\ \text{along}\\ \alpha \times \{-1\}\end{matrix} \cup \begin{matrix}\text{2-handles}\\ \text{along}\\ \beta \times \{1\}\end{matrix}$$

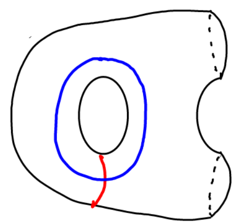


The examples from before:

A. $N(P) = (P \times I, \partial P \times I)$ has diagram $(P, \emptyset, \emptyset)$

B. If $Y = S^1 \times S^2$ with Heegaard diagram

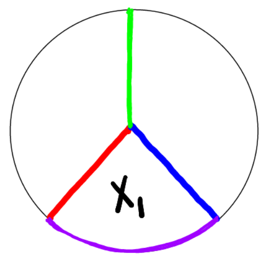


Then  is a sutured diag for $Y(P)$.

Back to relative trisections

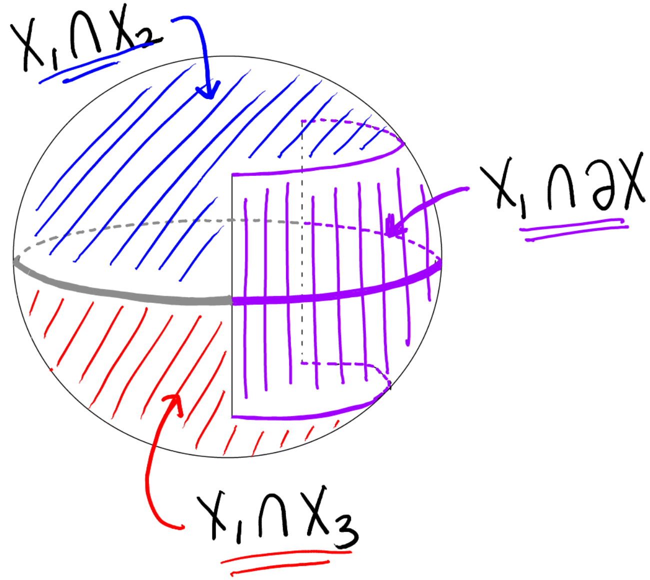
Def (A modification of Gay-Kirby) A relative trisection of a 4-mfd X with $\partial X \neq \emptyset$, $\pi_0 X = \{1\}$

is a decomposition $X = X_1 \cup X_2 \cup X_3$



$$X_i \cong \natural^k S^1 \times B^3$$

$$Y_k = \partial X_i \cong \#^k S^1 \times S^2$$
$$= \underline{(X_i \cap X_{i-1})} \cup \underline{(X_i \cap X_{i+1})} \cup \underline{(X_i \cap \partial X)}$$



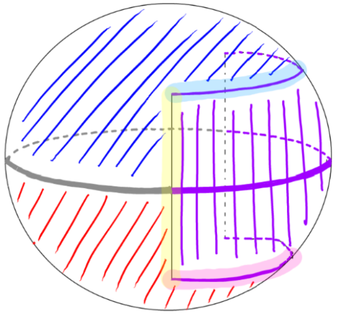
$$X_i \cap \partial X \cong N(P)$$

$(X_i \cap X_{i-1}) \cup (X_i \cap X_{i+1}) \cong$ a sutured Heegaard splitting of $Y_k(P)$

$X_1 \cap X_2 \cap X_3 = F$ a genus g surface with b bdry components

The induced structure on ∂X

In each piece:



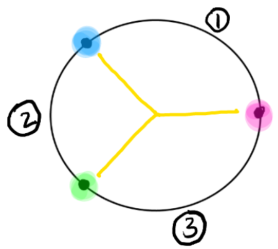
Interaction of the pieces (gluing):

hor. bdry: $\{1\} \times P$ in $X_i \cap \partial X$ is glued to $\{-1\} \times P$ in $X_{i+1} \cap \partial X$.

vert. bdry: $[0,1] \times \partial P$ in $X_i \cap \partial X$ is glued to $[-1,0] \times \partial P$ in $X_{i+1} \cap \partial X$ via $(t,x) \longrightarrow (-t,x)$

so:

$$\partial X = \bigcup_{i=1}^{3} X_i \cap \partial X \cong \bigcup_{i=1}^{3} I \times P \implies \partial X \cong I \times P \Big/ \begin{matrix} (-1,x) \sim (1,x) \\ (t,x) \sim (t',x) \text{ for } t,t' \in I,\ x \in \partial P \end{matrix}$$



an open book decomposition

Thm (Gay - Kirby) Let X be a smooth 4-mfd with non-empty and connected ∂X. For every OBD of ∂X, there exists a relative trisection of X.

"Open books can be filled with trisections".

A surprising fact about $\#^k S^1\times S^2$:

Let $Y_k = \#^k S^1\times S^2$ with:

Heegaard splitting $Y_k = H_- \cup H_+$,

" surface $S = H_- \cap H_+$

Let $L \subseteq S$ be a collection of curves s.t $\overline{S \setminus N(L)} = \Sigma \sqcup P$

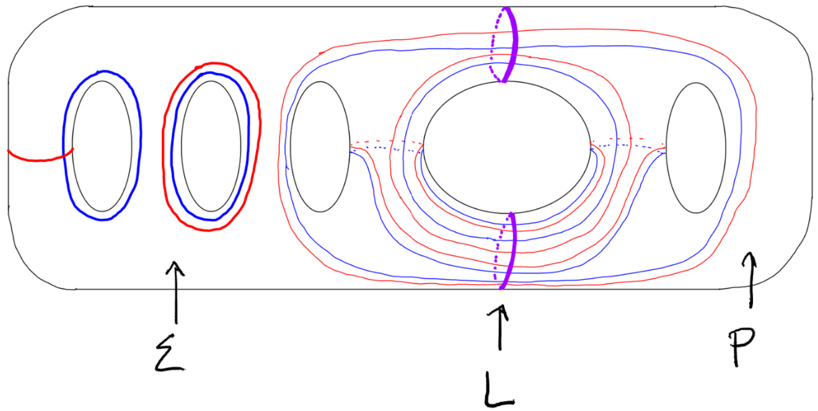
$$g(\Sigma) > g(\Sigma)$$

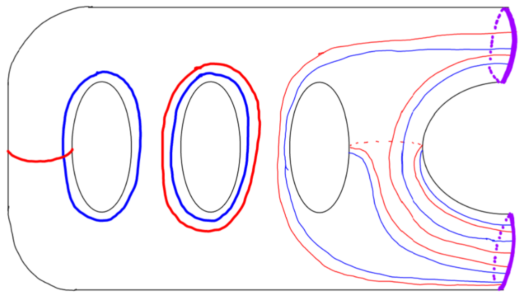
THM (Castro - Gay - P.) "There is a unique way of gluing $I\times P$ back to $Y_k(P)$"

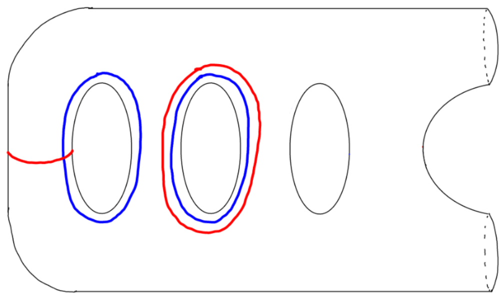
In other words,

- $Y_k(P)$ completely determines $\#^k S^1\times S^2$
- A sutured Heegaard diagram for $Y_k(P)$ completely determines a Heegaard diag. for $\#^k S^1\times S^2$

Ex: A genus 5 Heegaard diag. for $\#^4 S^1\times S^2$







Note: In this case $Y_4(P) \cong (S^1\times S^2)\#(I\times P)$

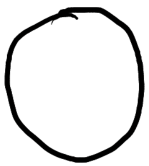
$1=5-4$

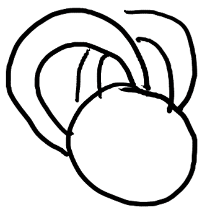
and in general $Y_k(P) \cong \left(\#^l S^1\times S^2\right)\#(I\times P)$ for some l.

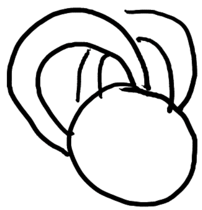
Def (Castro–Gay–P.) A relative trisection diagram is a tuple $(\Sigma, \alpha, \beta, \gamma)$ such that

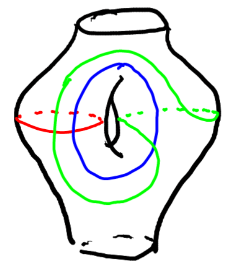
$$\left.\begin{array}{l}(\Sigma,\alpha,\beta)\\(\Sigma,\beta,\gamma)\\(\Sigma,\gamma,\alpha)\end{array}\right\}$$ are sutured Heegaard diagrams for $Y_k(P)$ (here $Y_k = \#_k S^1\times S^2$)

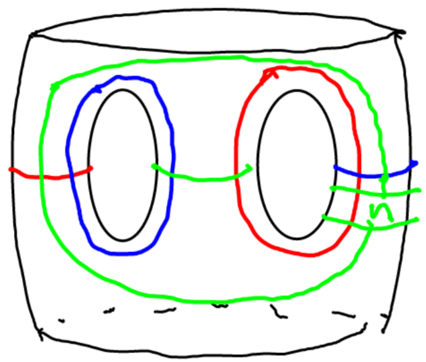
Examples:

B^4 

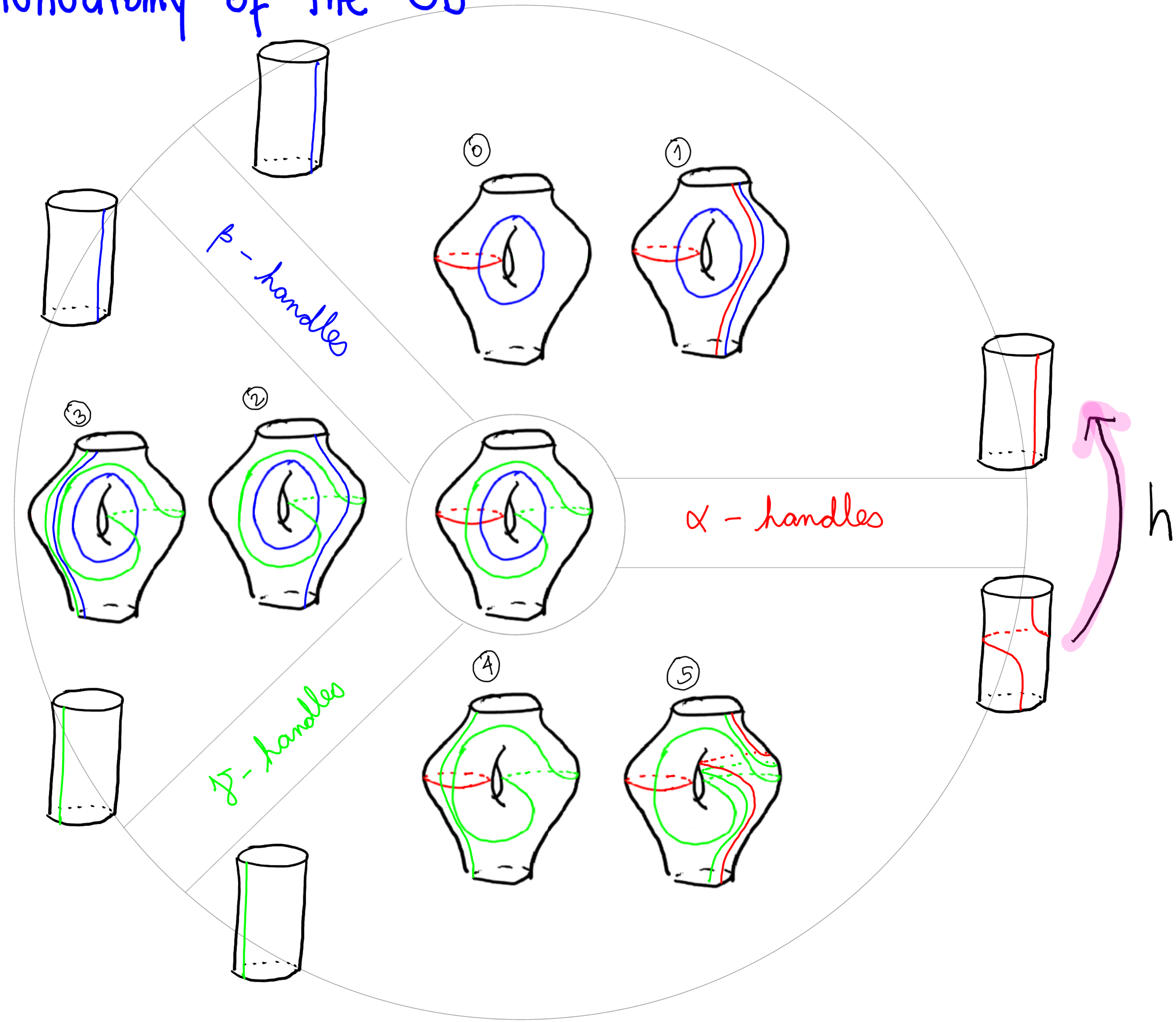
 $\times D^2$ has diagram



B^4 (Lekili) 

D^2 bundles over S^2 

Finding the monodromy of the OB



Thank you!